CALCULATING THE RADIATION COOLING OF SOLIDS

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We have derived an exact solution for the problem of the temperature field in solids cooled by radiation in the quasi-steady regime stage. For the initial cooling period we propose an approximate method of determining the temperature field, and this method is based on the principle of successive approximations. To facilitate the calculations, we have constructed engineering nomograms.

In a number of contemporary branches of engineering we encounter the problem of cooling solids by means of thermal radiation. Similar heat-transfer conditions are found in metallurgical processes of heat treatment, etc.

The heat flow from the cooled heat-transfer surface is formed in accordance with the Stefan–Boltzmann law

$$-\lambda\,(\mathrm{grad}\,T)_{\mathrm{cir}} = \sigma_a\,(T_{\mathrm{cir}}^4 - T_{\mathrm{w}}^4). \tag{1}$$

If $T_{sur} \gg T_w$, boundary conditions (1) assume the form

$$-\lambda\,(\mathrm{grad}\,T)_{\mathrm{cir}} = \sigma_a T_{\mathrm{cir}}^4. \tag{2}$$

The solution of problems relating to the steady-state process of heat transfer with radiative heat exchange involves substantial difficulties associated with the fact that the boundary condition (1) is nonlinear. In such case, as a rule, we use certain numerical methods of solutions that are possible by means of digital [1] or analog computers [2].

Here we propose a new approach to the analytical solution of radiation-cooling problems.

The mathematical formulation of this problem has the following form:

$$\frac{\partial\theta(X,\,\mathrm{Fo})}{\partial\,\mathrm{Fo}} = X^{-\nu}\frac{\partial}{\partial X}\left[X^{\nu}\frac{\partial\theta(X,\,\mathrm{Fo})}{\partial X}\right], \tag{3}$$

$$\frac{\partial\theta(1,\,\mathrm{Fo})}{\partial X} = -\,\mathrm{Sk}\,[\theta^4(1,\,\mathrm{Fo}) - \theta_{\mathrm{w}}^4], \tag{4}$$

$$\frac{\partial\theta(0,\,\mathrm{Fo})}{\partial X} = 0, \quad \theta(X,\,0) = 1. \tag{5}$$

Here $\theta = T/T_0$; $\mathrm{Sk} = \sigma_a T_0^3 R/\lambda$; T_0 is the initial temperature of the material; X is the relative coordinate; ν is a coefficient which characterizes the shape of the body and is numerically equal to 0, 1, and 2 in plane, cylindrical, and spherical coordinate systems, respectively. Applying the Laplace integral transform to (3)-(5) we find the solution of the problem in the images

$$\bar{\theta}(X,\,s) = \frac{1}{s} - U(1,\,s)\frac{\mathrm{ch}\sqrt{s}X}{\sqrt{s}\,\mathrm{sh}\sqrt{s}}, \quad \nu = 0, \tag{6a}$$

$$\bar{\theta}(X,\,s) = \frac{1}{s} - \begin{cases} U(1,\,s)\dfrac{I_0(\sqrt{s}X)}{\sqrt{s}I_1(\sqrt{s})}, & \nu = 1; \quad (6b) \\ U(1,\,s)\dfrac{\mathrm{sh}\sqrt{s}X}{X(\sqrt{s}\,\mathrm{ch}\sqrt{s} + \mathrm{sh}\sqrt{s})}, & \nu = 2; \quad (6c) \end{cases}$$

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$$\bar{\theta}(X, s) = \int_0^\infty \theta(X, \mathrm{Fo}) \exp(-s\,\mathrm{Fo})\, d\,\mathrm{Fo};$$

$$U(1, s) = \mathrm{Sk} \int_0^\infty [\theta^4(1, \mathrm{Fo}) - \theta_c^4] \exp(-s\,\mathrm{Fo})\, d\,\mathrm{Fo}.$$

Having completed the transition from the image to the original of the function, we derive the formal solution of the problem, i.e.,

$$\theta(X, \mathrm{Fo}) = 1 - \begin{cases} \mathrm{Sk}\int_0^{\mathrm{Fo}} [\theta^4(1, \eta) - \theta_w^4]\, d\eta + \frac{1 - 3X^2}{6}\mathrm{Sk}[\theta^4(1, \mathrm{Fo}) - \theta_w^4] - \\ \quad - \mathrm{Sk}\sum_{n=1}^{\infty} 2(-1)^{n+1}\frac{\cos\mu_n X}{\mu_n^2}\int_0^{\mathrm{Fo}} [\theta^4(1, \eta) - \\ \quad - \theta_w^4]' e^{-\mu_n^2(\mathrm{Fo}-\eta)}\, d\eta, \quad \nu = 0; \\ 2\,\mathrm{Sk}\int_0^{\mathrm{Fo}} [\theta^4(1, \eta) - \theta_w^4]\, d\eta + \frac{1 - 2X^2}{4}\mathrm{Sk}[\theta^4(1, \mathrm{Fo}) - \\ \quad - \theta_w^4] - \mathrm{Sk}\sum_{n=1}^{\infty} 2\frac{J_0(\mu_n X)}{\mu_n^2 J_1'(\mu_n)}\int_0^{\mathrm{Fo}} [\theta^4(1, \eta) - \\ \quad - \theta_w^4]' e^{-\mu_n^2(\mathrm{Fo}-\eta)}\, d\eta, \quad \nu = 1; \\ 3\,\mathrm{Sk}\int_0^{\mathrm{Fo}} [\theta^4(1, \eta) - \theta_w^4]\, d\eta + \frac{3 - 5X^2}{10}\mathrm{Sk}[\theta^4(1, \mathrm{Fo}) - \\ \quad - \theta_w^4] - \mathrm{Sk}\sum_{n=1}^{\infty} 2\frac{\frac{\sin\mu_n X}{\mu_n X}}{\mu_n^2\cos\mu_n}\int_0^{\mathrm{Fo}} [\theta^4(1, \eta) - \\ \quad - \theta_w^4]' e^{-\mu_n^2(\mathrm{Fo}-\eta)}\, d\eta, \quad \nu = 2. \end{cases} \tag{7}$$

If follows from (7) that the temperature at any cross section can be found only after we have determined the surface temperature. Assuming in (7) X = 1, we find the nonlinear functional equations for (1, Fo), i.e.,

$$\theta(1, \mathrm{Fo}) = 1 - \begin{cases} \mathrm{Sk}\int_0^{\mathrm{Fo}} [\theta^4(1, \eta) - \theta_w^4]\, d\eta - \frac{1}{3}\mathrm{Sk}[\theta^4(1, \mathrm{Fo}) - \theta_w^4] - \\ - \int_0^{\mathrm{Fo}} \sum_{n=1}^{\infty} \mathrm{Sk}\frac{2}{\mu_n^2}[\theta^4(1, \eta) - \theta_w^4]' e^{-\mu_n^2(\mathrm{Fo}-\eta)}\, d\eta, \quad \nu = 0; \\ 2\,\mathrm{Sk}\int_0^{\mathrm{Fo}} [\theta^4(1, \eta) - \theta_w^4]\, d\eta - \frac{1}{4}\mathrm{Sk}[\theta^4(1, \mathrm{Fo}) - \theta_w^4] - \\ - \int_0^{\mathrm{Fo}} \sum_{n=1}^{\infty} \mathrm{Sk}\frac{2}{\mu_n^2}[\theta^4(1, \eta) - \theta_w^4]' e^{-\mu_n^2(\mathrm{Fo}-\eta)}\, d\eta, \quad \nu = 1; \\ 3\,\mathrm{Sk}\int_0^{\mathrm{Fo}} [\theta^4(1, \eta) - \theta_w^4]\, d\eta - \frac{1}{5}\mathrm{Sk}[\theta^4(1, \mathrm{Fo}) - \theta_w^4] - \\ - \int_0^{\mathrm{Fo}} \sum_{n=1}^{\infty} \mathrm{Sk}\frac{2}{\mu_n^2}[\theta^4(1, \eta) - \theta_w^4]' e^{-\mu_n^2(\mathrm{Fo}-\eta)}\, d\eta, \quad \nu = 2. \end{cases} \tag{8}$$

It is not difficult to demonstrate – as is done, for example, in [3] – that the classical method of successive approximations converges in the solution of a nonlinear functional equation of the form of (8). However, in

TABLE 1. Roots of Eq. 10

θ_w	$\frac{Sk}{3+\nu}$				
	0,1	0,25	0,5	0,75	1,0
0	0,9264	0,8623	0,7978	0,7559	0,7345
0,25	0,9268	0,8628	0,7985	0,7570	0,7260
0,50	0,9310	0,8710	0,8138	0,7755	0,7489
0,75	0,9503	0,9087	0,8708	0,8486	0,8334

TABLE 2. Values of the Function $\Phi_1(\theta,\ Sk/3+\nu)$

θ	$\frac{Sk}{3+\nu}$				
	0,1	0,25	0,5	0,75	1,0
0,1	334,25	335,64	337,94	340,241	342,543
0,2	42,3043	43,261	44,855	46,4496	48,044
0,3	12,8276	13,550	14,7539	15,9579	17,1619
0,4	5,5745	6,1243	7,0406	7,9569	8,8732
0,5	2,9439	3,3598	4,0530	4,7461	5,4393
0,6	1,7483	2,0563	2,5697	3,0829	3,5963
0,7	1,1145	1,3286	1,6854	2,0421	2,3989
0,8	0,7403	0,8742	1,0974	1,3206	1,5438
0,9	0,4994	0,5626	0,6680	0,7734	0,8788
0,95	0,4093	0,4401	0,4914	0,5427	0,59406
0,99	0,3445	0,3505	0,3604	0,3703	0,3803

TABLE 3. Results from the Calculation of the Time and Temperature for an Unbounded Plate

θ (1, Fo)	Fo		Fo	θ (0, Fo)	
	after [12]	digital computer data		after [11]	digital computer data
0,7	0,3912	0,38	0,3	0,90475	0,901
0,6	0,9807	0,98	0,5	0,8148	0,813
0,5	1,9696	2,0	1	0,6774	0,676
0,4	3,9696	4,07	2	0,5169	0,517
0,3	9,1035	9,1	5	0,3903	0,389
0,2	29,17	29	10	0,2975	0,301
0,1	224,56	224	50	0,1656	0,166

order to derive theoretical relationships more convenient from the practical standpoint, within whose structure we find no sums of infinite series, the solution of the formulated problem is achieved separately for large and small values of the Fo number.

1. The Quasi-Study. We know [4] that on elapse of a certain period of time Fo > Fo_1 the body at whose surface the heat flow is variable enters the stage of an ordered regime. From the physical standpoint this suggests the self-similarity of the process, and this is expressed in that the distribution of the temperature through the cross section with the passage of time is identical in nature. From the mathematical standpoint this indicates that the last series in (7) and in (8) becomes relatively small in comparison with the remaining terms, so that without introduction of some substantial error it is possible to neglect the entire series. With consideration of this circumstance, we can derive the following closed solution for the integral equation (8):

$$\mathrm{Sk}(\nu+1)\mathrm{Fo} = \left[\frac{1}{4\theta_w^3}\ln\left|\frac{\theta_w+\theta(1,\mathrm{Fo})}{\theta_w-\theta(1,\mathrm{Fo})}\right| + \frac{1}{2\theta_w^3}\operatorname{arctg}\frac{\theta(1,\mathrm{Fo})}{\theta_w} - \frac{\mathrm{Sk}}{3+\nu}\ln|\theta_w^4-\theta^4(1,\mathrm{Fo})|\right]$$
$$-\left[\frac{1}{4\theta_w^3}\ln\left|\frac{\theta_w+\theta_*}{\theta_w-\theta_*}\right| + \frac{1}{2\theta_w^3}\operatorname{arctg}\frac{\theta_*}{\theta_w} - \frac{\mathrm{Sk}}{3+\nu}\ln|\theta_w^4-\theta_*^4|\right]$$
$$= \Phi\left(\theta,\ \theta_w\ \frac{\mathrm{Sk}}{3+\nu}\right) - \Phi\left(\theta_*,\ \theta_w\ \frac{\mathrm{Sk}}{3+\nu}\right), \qquad (9)$$

where θ_* is found from the formula

$$\theta_* = -\frac{\sqrt{2y}}{2} + \sqrt{\frac{\frac{3+\nu}{\mathrm{Sk}}}{\sqrt{8y}} - \frac{y}{2}};$$

$$y = \sqrt[3]{\left(\frac{3+\nu}{4\,\mathrm{Sk}}\right)^2 + \sqrt{\left(\frac{3+\nu}{4\,\mathrm{Sk}}\right)^4 + \left(\frac{\frac{3+\nu}{\mathrm{Sk}} + \theta_W^4}{3}\right)^3}} + \sqrt[3]{\left(\frac{3+\nu}{4\,\mathrm{Sk}}\right)^2 - \sqrt{\left(\frac{3+\nu}{4\,\mathrm{Sk}}\right)^4 + \left(\frac{\frac{3+\nu}{\mathrm{Sk}} + \theta_W^4}{3}\right)^3}}. \quad (10)$$

Now that we know the temperature at the surface of the body, on the basis of (7) we can calculate the temperature at any point on the body from the formula

$$\theta(X, \mathrm{Fo}) = \theta(1, \mathrm{Fo}) + \frac{\mathrm{Sk}}{2}[\theta^4(1, \mathrm{Fo}) - \theta_w^4](1 - X^2). \quad (11)$$

When $\theta_W = 0$ (boundary condition (2)) Eq. 9 assumes the form

$$\mathrm{Sk}(\nu+1)\mathrm{Fo} = \left[\frac{1}{3}\theta^{-3}(1, \mathrm{Fo}) - \frac{\mathrm{Sk}}{3+\nu}\ln\theta^4(1, \mathrm{Fo})\right] - \left(\frac{1}{3}\theta_*^{-3} - \frac{\mathrm{Sk}}{3+\nu}\ln\theta_*^4\right) = \Phi_1\left(\theta, \frac{\mathrm{Sk}}{3+\nu}\right) - \Phi_1\left(\theta_*, \frac{\mathrm{Sk}}{3+\nu}\right), \quad (12)$$

while θ_* is found from (10) when $\theta_W^4 = 0$.

We note that the problem of determining the instant at which a specified surface temperature is attained involves no particular difficulties, since the Fo number is not related by a functional relationship. To facilitate the execution of the calculations associated with the determination of the temperature $\theta(1, \mathrm{Fo})$, Tables 1 and 2 give the values of the function $\Phi(\theta, \mathrm{Sk}/3+\nu)$ and of θ_*, respectively. Table 3 shows the results from the calculation of the time and the temperature field at the surface and at the center of a flat body ($\nu = 0$) and there is a comparison with the data from the numerical integration carried out on the M-20 computer for the case Sk = 1.5, $\theta_W = 0$.

Comparison of the calculation results confirms one more time that the derived solutions (9) and (11) are analytically exact for the quasi-steady state.

2. Initial Period. We will demonstrate the method of solution on the example of the cooling of an unbounded plate. Expanding $1/\mathrm{sh}\sqrt{s}$ in (6a) in a power series, limiting ourselves to the first member of the expansion, and returning to the originals of the function, and using the Borel theorem, we can derive the following formal solution:

$$\theta(X, \mathrm{Fo}) = 1 - \mathrm{Sk}\int_0^{\mathrm{Fo}}[\theta^4(1, \mathrm{Fo}-\eta) - \theta_w^4]\frac{e^{-\frac{(1-X)^2}{4\eta}}}{\sqrt{\pi\eta}}d\eta - \mathrm{Sk}\int_0^{\mathrm{Fo}}[\theta^4(1, \mathrm{Fo}-\eta) - \theta_w^4]\frac{e^{-\frac{(1+X)^2}{4\eta}}}{\sqrt{\pi\eta}}d\eta. \quad (13)$$

Assuming in (13) that X = 1, we find that the surface temperature satisfies the nonlinear integral Volterra equation of the second kind

$$\theta(1, \mathrm{Fo}) = 1 - \mathrm{Sk}\int_0^{\mathrm{Fo}}[\theta^4(1, \mathrm{Fo}-\eta) - \theta_w^4]\frac{d\eta}{\sqrt{\pi\eta}} - \mathrm{Sk}\int_0^{\mathrm{Fo}}[\theta^4(1, \mathrm{Fo}-\eta) - \theta_w^4]\frac{e^{-1/\eta}}{\sqrt{\pi\eta}}d\eta,$$

Let us evaluate the last integral in (14)

$$\max[\theta^4(\eta) - \theta_w^4] = f(\eta) \leqslant M$$

for $\eta \in (0, \mathrm{Fo})$. Then

$$\frac{\mathrm{Sk}}{\sqrt{\pi}}\int_0^{\mathrm{Fo}}[\theta^4(1, \mathrm{Fo}-\eta) - \theta_w^4]\frac{e^{-1/\eta}}{\sqrt{\eta}}d\eta \leqslant \frac{M}{\sqrt{\pi}}\int_0^{\mathrm{Fo}} e^{-1/\eta}\eta^{-1/2}d\eta = M\left[\frac{2\sqrt{\mathrm{Fo}}\,e^{-1/\mathrm{Fo}}}{\sqrt{\pi}} - 2\left(1 - \mathrm{erf}\frac{1}{\sqrt{\mathrm{Fo}}}\right)\right],$$

where $\mathrm{erf}\,1/\sqrt{\mathrm{Fo}}$ is the error integral.

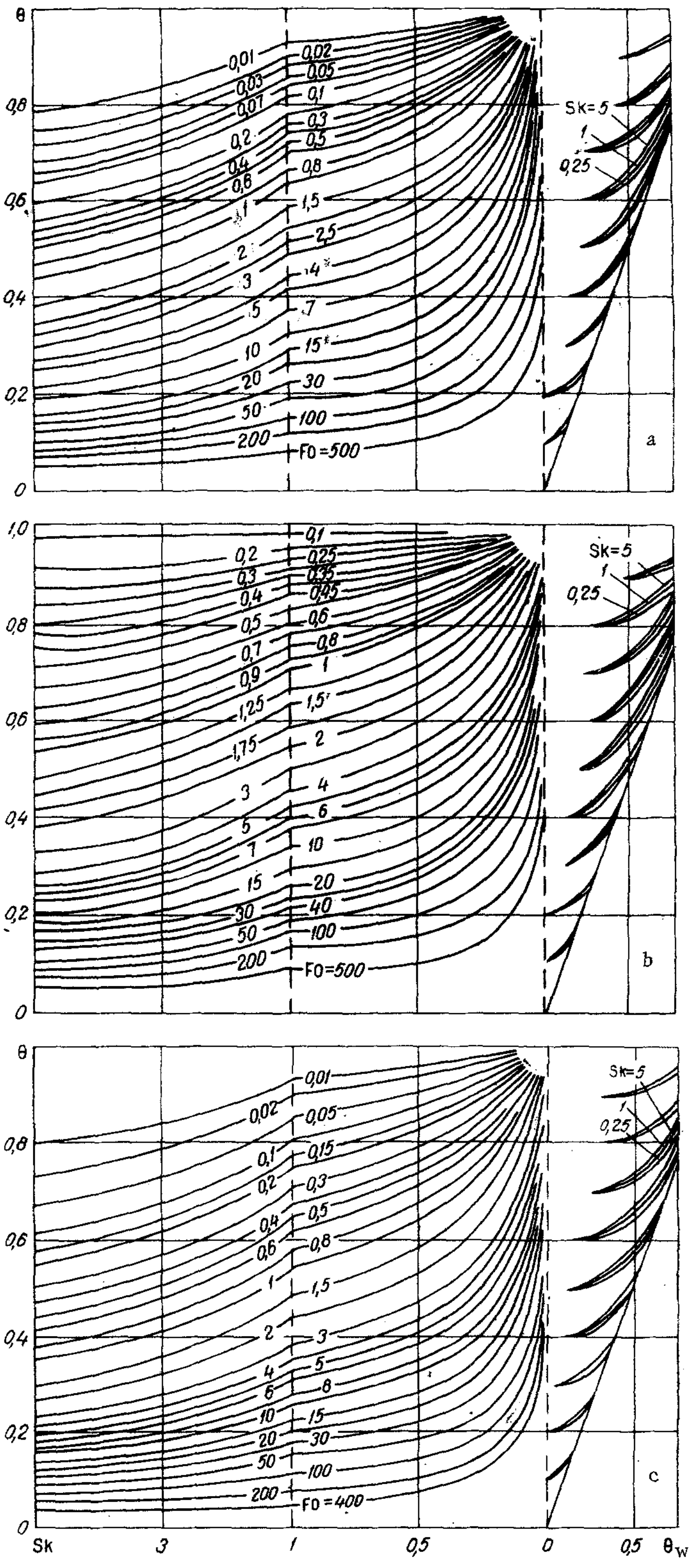

Fig. 1. Nomogram for the calculation of the relative temperature at the surface (a) and at the center (b) of an unbounded plate, at the surface (c) and at the axis (d): (continued on next page) of an infinite cylinder.

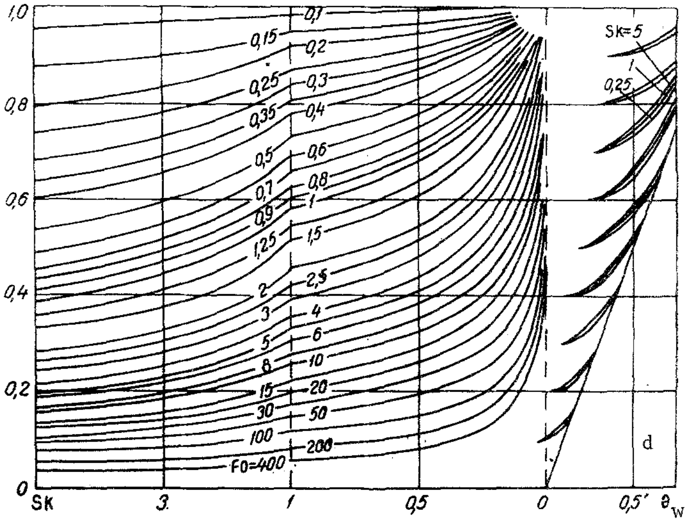

Fig. 1. (Continued)

Thus, for small values of Fo this integral is extremely insignificant and in an approximate solution it can be neglected without introducing significant error. Indeed, when Fo = 0.25 the quantity in the brackets is equal to 0.0009, while when Fo = 0.35 it is 0.0045.

Bearing in mind that the kernel of the integral equation is not unique, to find the solution of that equation we can employ the classical method of successive approximations. As the zeroth approximation it is advisable to choose the free term in (14). Having substituted the derived law governing the variation of surface temperature into (13) and having performed the simple integration, we can write the final expression with respect to the calculation of the temperature field for as high an order of approximation as one pleases.

The numerical calculations show that in all cases a reasonable eror is guaranteed, as a rule, if we restrict ourselves to the second–third approximation. Carrying out analogous mathematical calculations, we can derive a solution by the above method for the problem of radiation cooling of an unbounded cylinder or sphere.

There is some technical interest in using the proposed solution methods to develop engineering nomograms which would make it possible to speed up the calculation of the temperature field in a radiation-cooling regime. The figures show such nomograms for the calculation of the temperature; these calculations are, respectively, for the temperature at the surface and at the center of an unbounded plate and cylinder. It must be stressed that the proposed nomograms encompass a rather broad range of regime parameters and are more universal, unlike the calculation graphs given in [1, 2].

We note that this method of solving nonlinear heat-conduction problems is applicable even when the body contains an internal heat source of constant power.

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